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By H. N. Hill, M. ASCE, and J. W. Clark, Jun. ASCE

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PAPERS

LATERAL BUCKLING OF ECCENTRICALLY LOADED I-SECTION COLUMNS

By H. N. Hill,[1] M. ASCE, and J. W. Clark,[2] Jun. ASCE

Synopsis

This paper describes an experimental investigation of the behavior of I-section columns loaded eccentrically in the plane of the web. All the specimens tested failed by combined bending and twisting at stresses within the elastic range of the material.

The test results confirm theoretical work that has been done on the subject and indicate that a previously proposed design formula to cover this type of failure is unconservative.

Introduction

The ultimate strength of a member under an axial compressive force is reduced if the member is at the same time required to resist transverse bending moments. Conversely the ability to resist transverse bending is impaired by the presence of an axial compressive force. Consequently, in designing a member to be subjected simultaneously to axial compression and transverse bending, it is necessary to reduce the allowable end load and the allowable bending moment below the values which would be permitted if either occurred alone.

A member under end load and bending may fail in either of two general ways: (1) By excessive bending in the plane of the applied bending moment; or (2) by lateral buckling, involving twisting of the member and bending in a direction other than the plane of the applied bending moment. This second type of failure is the subject of this paper.

A design formula to cover the case of lateral buckling of a member under simultaneous end load and bending was published in 1938.[3] The formula was

Note.—Written comments are invited for publication; the last discussion should be submitted by March 1, 1951.

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[3] "Structural Aluminum Handbook," Aluminum Co. of America, Pittsburgh, Pa., 1938.

based on a theoretical solution of the problem. At the time a test program was planned to supply experimental evidence of the adequacy of the formula. The test program was considerably delayed and in the meantime a more rigorous theoretical solution of the problem was published.[4,5] On completion of the test program it seemed advisable to reexamine the design formula in the light of the test results and the more recent theoretical treatment.

Object of the Investigation

The object of the investigation described in this paper was to study the behavior of I-section members under simultaneous end load and bending in the plane of the web. For convenience in testing, the specimens were bent by applying end loads with eccentricity in the plane of the web. The specimens tested were therefore eccentrically loaded columns.

Description of Specimens

The average measured dimensions for each of the extruded I-section specimens employed in this investigation and section elements based on these dimensions are given in Table 1. The areas corresponding to the measured dimensions agreed with areas computed on the basis of the weight and specific gravity of the specimens within 1% in all cases.

The specimens were of aluminum alloy 27S-T6. At the time the tests were started, this alloy was commonly used in structural applications.[6] It is no longer a standard product, however, having been largely replaced in the structural field by alloy 14S-T6. Tensile tests (on standard specimens[7] $\frac{1}{2}$ in. wide) and pack compressive tests[8] were made, the specimens being cut in a longitudinal direction from the flanges of the I-sections. The results of these tests are given in Table 1. The properties are typical of 27S-T6 alloy. The compressive modulus of elasticity of this alloy is 10,700,000 lb per sq in. and Poisson's ratio is $\frac{1}{3}$.

The measured crookedness, listed in Table 1, was obtained in most cases by placing the specimens on a flat surface and measuring the maximum clearance between the surface and the flanges of the specimen by a feeler gage. The crookedness of specimen 13 and of specimen 10, after straightening, was obtained by supporting the specimen as a beam at the ends and measuring the distance to a flat surface at points along the length with a dial gage. The specimens were then turned over and the measurements repeated to eliminate the effect of deflection caused by the weight of the specimens.

Description of Tests

The columns were tested in a 40,000-lb Amsler machine, using trunnion type ball bearing heads of 10,000-lb capacity. These heads provided a hinged

[4] "The Buckling of Compressed Bars by Torsion and Flexure," by J. N. Goodier, *Bulletin No. 27*, Cornell Univ. Eng. Experiment Station, Ithaca, N. Y., 1941.

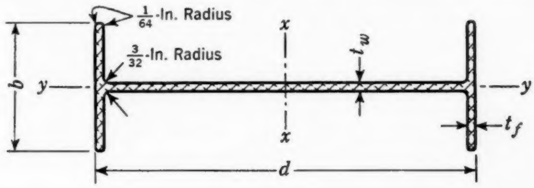
[5] "Flexural-Torsional Buckling of Bars of Open Section," by J. N. Goodier, *Bulletin No. 28*, Cornell Univ. Eng. Experiment Station, Ithaca, N. Y., 1942.

[6] "Light-Weight Structural Alloys," by Zay Jeffries, C. F. Nagel, and R. T. Wood, in "Structural Application of Steel and Light-Weight Alloys," *Transactions*, ASCE, Vol. 102, 1937, p. 1279, Table 7(*b*).

[7] "Standard Methods of Tension Testing of Metallic Materials," *1946 Book of A.S.T.M. Standards Including Tentatives*, Pt. I-B, 1946, A.S.T.M. designation: E8-46, p. 297, Fig. 2.

[8] "The Pack Method for Compressive Tests of Thin Specimens of Materials Used in Thin-Wall Structures," by C. S. Aitchison and L. B. Tuckerman, *Technical Report No. 649*, National Advisory Committee for Aeronautics, Washington, D. C., 1939.

TABLE 1.—ELEMENTS OF SECTIONS AND MECHANICAL PROPERTIES OF 27S-T6 EXTRUDED I-SECTION SPECIMENS



Specimen	Depth, d, in inches	Flange width, b, in inches	Thickness, in Inches		Area, in square inches	Moment of Inertia, in Inches[4]	
			Flange, t_f	Web, t_w		I_x	I_y
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	3.990	1.383	0.0960	0.0775	0.5676	1.388	0.04248
2	3.991	1.383	0.0958	0.0787	0.5716	1.392	0.04239
3	3.993	1.382	0.0963	0.0763	0.5638	1.387	0.04253
4	3.998	1.381	0.0963	0.0780	0.5696	1.390	0.04246
5	4.001	1.387	0.0972	0.0817	0.5882	1.430	0.04340
7	3.994	1.383	0.0963	0.0768	0.5660	1.391	0.04261
9	3.996	1.384	0.0972	0.0817	0.5873	1.424	0.04315
10	3.993	1.382	0.0963	0.0780	0.5702	1.395	0.04255
11	3.992	1.382	0.0962	0.0788	0.5730	1.397	0.04251
12	3.993	1.383	0.0959	0.0768	0.5647	1.386	0.04240
13	3.993	1.382	0.0962	0.0778	0.5692	1.393	0.04246

TABLE 1.—(*Continued*)

Specimen	Torsion factor, J, in inches[4]	Tensile strength, in pounds per square inch	Yield Strength, in Pounds per Square Inch[a]		Crookedness,[b] in Inches	
			Tension	Compression	West flange	East flange
(1)	(9)	(10)	(11)	(12)	(13)	(14)
1	0.001435	63,700	55,000	55,000	0.012 S	0.013 N
2	0.001459	64,680	56,000	55,500	....	....
3	0.001414	63,270	54,900	55,300	0.006 S	0.006 S
4	0.001453	63,950	56,000	56,000	....	....
5	0.001576	63,270	54,900	55,500	0.006 S	0.023 N
7	0.001427	63,300	54,800	55,500	0.045 N	0.016 N
9	0.001574	63,270	54,900	55,500	0.018 N	0.014 N
10	0.001455	63,950	56,000	56,000	0.022 S[c]	0.016 S[c]
11	0.001472	64,680	56,000	55,500	0.017 S	0.016 S
12	0.001416	61,820	53,000	54,200	0.026 N	0.018 S
13	0.001447	63,510	52,900	52,300	0.095 S	0.032 N

[a] Set = 0.2%. [b] N and S indicate maximum bow to the north and south, respectively. [c] These are the values before straightening. After straightening the "crookedness" was reduced to 0.008 in. and 0.006 in., for the west flange and east flange, respectively.

condition in the direction of the axis in the web (with the center of rotation in the planes of the ends of the specimen) and a flat-end condition in the direction of the axis normal to the web. The test setup is illustrated in Fig. 1, with specimen 10, Table 1, in position for testing at an eccentricity of 1 in.

Columns having lengths of 40 in., 60 in., 70 in., and 100 in. were tested, with eccentricities of load in the plane of the web of 0 in., $\frac{1}{4}$ in., $\frac{1}{2}$ in., and 1 in. Eccentricity was measured with reference to concentric circles inscribed on the

faces of the heads. The maximum eccentricity was within the core of the section—that is, within a region such that there was no tendency to develop tensile stresses between the specimen and the platens. In this region of small eccentricities (that is, large ratios of end load to Euler load), the maximum safe stress is appreciably lower than that for bending alone.

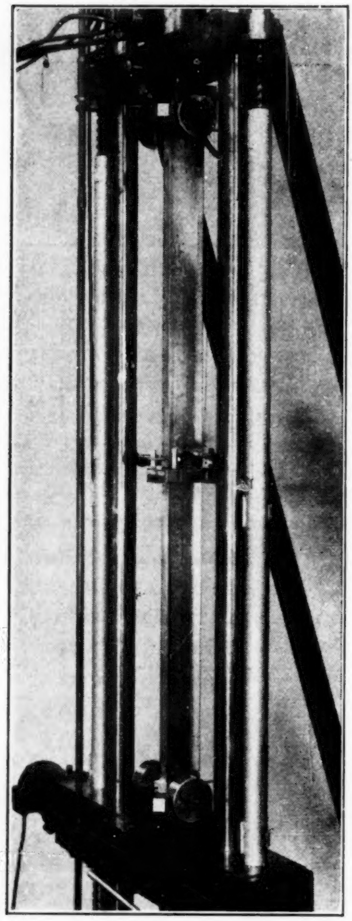


FIG. 1.—ECCENTRICALLY LOADED COLUMN SPECIMEN 10 IN POSITION FOR TESTING

The stresses developed in the columns were measured with four Huggenberger tensometers (type A: Nominal magnification 1,200 and gage length 1 in.) clamped to the edges of the flanges at the centers of the specimens as shown in Fig. 1.

Deflections were measured with the aid of mirrored scales and taut wires. In the tests in which the load was applied with no eccentricity, deflections perpendicular to the web were measured on both flanges, 2 in. below the center. When an eccentric loading was used, the deflection perpendicular to the web was measured on the flange with the greater compression, and the deflection at the center in the direction of the web was also measured.

In all cases, loading was continued in increments until a definite maximum had been reached, as indicated by a falling off of the load despite continued movement of the head of the testing machine.

The 40-in. specimens were each tested only once since failure was accompanied by some permanent set. Failure was elastic in the remaining columns so that the same specimen could be tested at different eccentricities. Specimen 10 was retested after having been straightened by loading the specimen as a beam perpendicular to the plane of the web.

RESULTS AND DISCUSSION

Table 2 lists the loads at failure. Typical plots of the measured deflections and the stresses corresponding to the measured strains are shown in Figs. 2 and 3 for specimen 11. Deflection curves, both in the plane of the web and normal to this plane, are shown in Fig. 4 for all the eccentricities of load under which this specimen was tested. It can be seen from the curves of deflection perpendicular to the web (Fig. 4(b)) that failure in each case was by lateral buckling. This was true of all specimens tested.

TABLE 2.—LATERAL BUCKLING OF ECCENTRICALLY LOADED I-SECTION COLUMNS

Specimen	Length, L, in inches	Eccentricity, e, in inches	Loads, P, at failure, in pounds	Computed Critical Column Load, in Pounds		Critical Bending Moment	Computed Loads at Failure, in Pounds			Percentage Variation[b]		
				$K=0.5$	$K=$test value	$K=$test value	Handbook formula[a] (Eq. 3)	Goodier Solution: Before adjustment	Goodier Solution: Adjusted for deflection	Handbook formula[a] (Eq. 3)	Goodier Solution: Before adjustment	Goodier Solution: Adjusted[c]
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
Coefficient $K=0.5175$—												
1	39 15/16	0	10,500	11,250	10,500	....	....	....	....	....	....	....
2	39 31/32	1/4	10,250	11,205	10,458	21,840	10,312	10,140	10,076	+ 0.6	−1.1	−1.7
3	40	1/2	9,230	11,229	10,480	21,842	9,938	9,446	9,295	+ 7.7	+2.3	+0.7
4	40	1	7,750	11,210	10,463	21,822	8,772	7,970	7,758	+13.2	+2.8	+0.1
5	39 31/32	1	8,140	11,472	10,707	22,480	8,993	8,192	7,997	+10.5	+0.6	−2.0
Coefficient $K=0.5046$—												
7	60	0	4,910	5,000	4,910	....	....	....	....	....	....	....
		1/4	4,780		4,910	10,949	4,850	4,793	4,768	+ 1.5	+0.3	−0.3
		1/2	4,380		4,910	10,949	4,685	4,517	4,450	+ 7.0	+3.1	+1.6
		1	3,645		4,910	10,949	4,191	3,879	3,773	+15.0	+6.4	+3.5
Coefficient $K=0.5079$—												
9	60 1/32	0	4,900	5,057	4,900	....	....	....	....	....	....	....
		1/4	4,835		4,900	11,064	4,841	4,790	4,766	+ 0.1	−0.9	−1.4
		1/2	4,540		4,900	11,064	4,680	4,526	4,462	+ 3.1	−0.3	−1.7
		1	3,840		4,900	11,064	4,195	3,901	3,798	+ 9.2	+1.6	−1.1
Before Straightening, Coefficient $K=0.5341$—												
10	69 15/16	0	3,220	3,674	3,220	....	....	....	....	....	....	....
		1/4	3,170		3,220	7,632	3,185	3,158	3,146	+ 0.5	−0.4	−0.8
		1/2	3,005		3,220	7,632	3,088	3,005	2,971	+ 2.8	0	−1.1
		1	2,610		3,220	7,632	2,790	2,621	2,561	+ 6.9	+0.4	−1.9
After Straightening, Coefficient $K=0.5248$—												
10	69 15/16	0	3,335	3,674	3,335	....	....	....	....	....	....	....
		1/4	3,280		3,335	7,861	3,298	3,270	3,257	+ 0.5	−0.3	−0.7
		1/2	3,100		3,335	7,861	3,197	3,109	3,072	+ 3.1	+0.3	−0.9
		1	2,680		3,335	7,861	2,886	2,708	2,643	+ 7.7	+1.0	−1.4
Coefficient $K=0.5308$—												
11	69 31/32	0	3,255	3,668	3,255	....	....	....	....	....	....	....
		1/4	3,225		3,255	7,714	3,220	3,193	3,180	− 0.2	−1.0	−1.4
		1/2	3,055		3,255	7,714	3,122	3,038	3,003	+ 2.2	−0.6	−1.7
		1	2,655		3,255	7,714	2,820	2,651	2,589	+ 6.2	−0.2	−2.5
Coefficient $K=0.5147$—												
12	100	0	1,690	1,791	1,690	....	....	....	....	....	....	....
		1/4	1,645		1,690	4,525	1,675	1,668	1,663	+ 1.8	+1.4	+1.1
		1/2	1,570		1,690	4,525	1,635	1,610	1,594	+ 4.1	+2.5	+1.5
		1	1,375		1,690	4,525	1,503	1,444	1,411	+ 9.3	+5.0	+2.6
Coefficient $K=0.5201$—												
13	100 1/16	0	1,655	1,791	1,655	....	....	....	....	....	....	....
		1/4	1,650		1,655	4,479	1,641	1,634	1,630	− 0.5	−1.0	−1.2
		1/2	1,610		1,655	4,479	1,602	1,578	1,564	− 0.5	−2.0	−2.9
		1	1,435		1,655	4,479	1,475	1,419	1,389	+ 2.8	−1.1	−3.2

[a] "Alcoa Structural Handbook," Aluminum Co. of America, Pittsburgh, Pa., 1945, p. 58. [b] Percentage variation of computed values from the actual loads at failure. [c] The Goodier solution adjusted for deflection.

Curves of calculated bending deflection in the plane of the web are shown in Fig. 4(a). Deflections were computed from the formula:[9]

$$\delta = e\left(\sec\frac{\pi}{2}\sqrt{\alpha} - 1\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

[9] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 13.

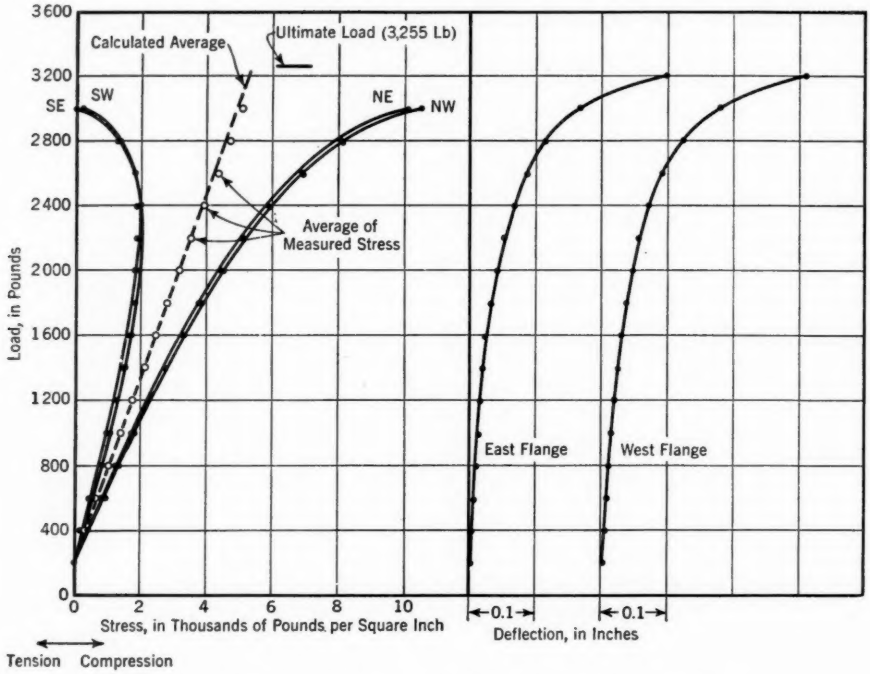


Fig. 2.—Measured Stresses and Deflections in Specimen 11 (See Table 2), with No Eccentricity

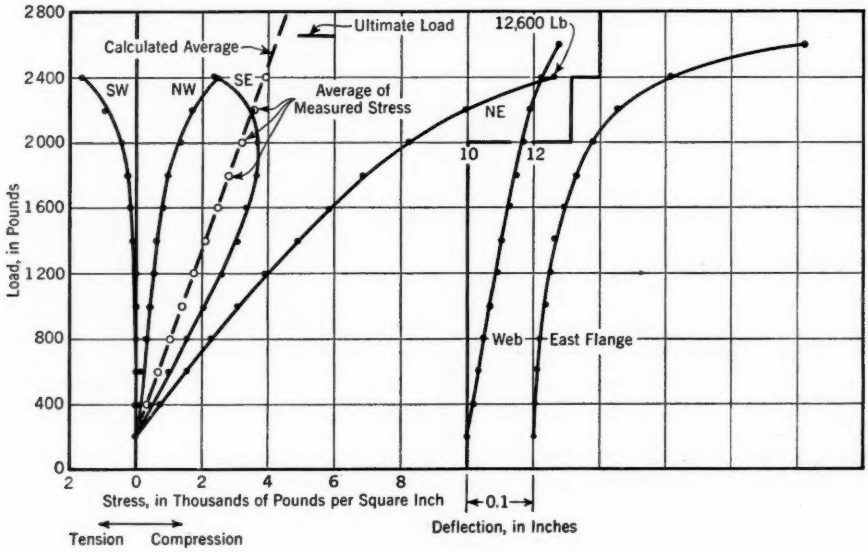


Fig. 3.—Measured Stresses and Deflections in Specimen 11 (See Table 2), with 1-In. Eccentricity

in which δ is the deflection at the center in the plane of the web, in inches; e is the eccentricity of load, in inches; and α is the ratio of load on the column to the Euler critical load for a pinned-end column. Figs. 5 and 6 show representative comparisons between measured and calculated bending stresses in the plane of the web. Bending stresses were computed by the formula:

$$f_b = \frac{P\,y}{I}\,(e + \delta) \qquad (2)$$

in which f_b is the maximum bending stress, in pounds per square inch; P is the load on the column, in pounds; and y is the distance from the neutral axis to the point of attachment of the tensometers, in inches. There was good agreement between the test results and the calculated values of bending stress and deflection in the plane of the web. This agreement can be considered as indicating that the eccentricities of loading were very close to the desired values.

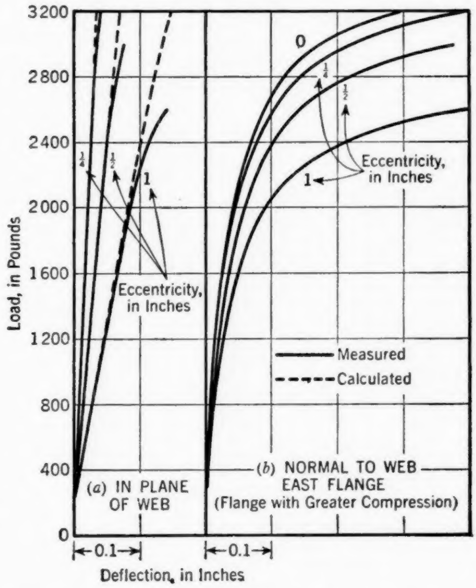


Fig. 4.—Deflections of Web and East Flange; Specimen 11

The critical load has been computed for each specimen tested with zero eccentricity, using the section elements given in Table 1. Complete end fixity was assumed because the specimens all failed by bending laterally, in which direction they functioned as flat-end columns. The computed critical load values are listed in Table 2 for comparison with the test results. The test loads are slightly smaller than the computed critical loads in every case, indicating that the flat-end condition of the tests resulted in less than complete fixity. The fixity coefficients (K-values) corresponding to the test results were computed and are shown in Table 2. These values range from 0.505 to 0.531 (the value for complete fixity being 0.5; and that for pinned ends, 1.0). Variations in the value of the fixity coefficient for the different tests are probably associated with initial crookedness, and with deviations of the ends of the specimens from the desired condition of being parallel and normal to the axis. The ends of these specimens were finished in a milling machine and checked for parallelism by measuring the length at the four corners with an outside caliper incorporating a 0.001-in. dial indicator. The range of fixity factors encountered is typical of previous experience on columns so machined. In more recent investigations more nearly perfect end conditions have been obtained by turning the ends of the specimens on centers or on an arbor in a lathe.

A working specification[10] for allowable bending stress in the presence of an axial load, for the case involving lateral buckling, permits a maximum bending stress (compression) on the extreme fiber, at or near the center of the unsupported length, in addition to direct compression, expressed by

$$\frac{Mc}{I} = f_a \sqrt{1 - \frac{\frac{P}{A}}{f'_c}} \qquad (3)$$

In Eq. 3 (in pounds per square inch), $\frac{P}{A}$ is the average compressive stress on the cross section of the member produced by column load; f_a is the allowable

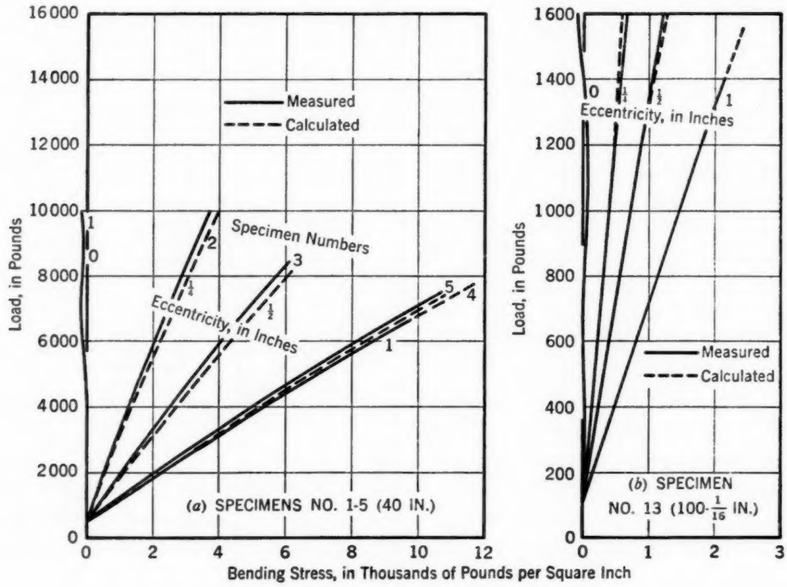


FIG. 5.—BENDING STRESSES IN THE PLANE OF THE WEB

compressive working stress for a member considered as a beam; and f'_c is the allowable working stress for a member considered as a column tending to fail in the direction normal to the plane of the bending forces. Eq. 3 was based on the theoretical solution given by S. Timoshenko[11] for lateral buckling of deep rectangular beams, under simultaneous end load and pure bending. This solution can be expressed as

$$\frac{P}{P'} + \left(\frac{M}{M'}\right)^2 = 1 \qquad (4)$$

[10] "Alcoa Structural Handbook," Aluminum Co. of America, Pittsburgh, Pa., 1945, p. 58.

[11] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 243.

in which P is the axially applied end load, in pounds; P' is the critical end load for buckling in the direction of least stiffness, in pounds; M is the bending moment applied in the direction of greatest stiffness, simultaneously with the end load P, in inch-pounds; and M' is the critical bending moment for the beam under pure bending alone, in inch-pounds. Bruce Johnston,[12] M. ASCE, has derived the same relationship for I-section members. Eq. 3 is derived from Eq. 4 by applying safety factors and expressing the end load and the bending in terms of unit stresses.

For the purpose of comparing the results of these tests with the foregoing treatment, the maximum load values have been expressed as ratios $\frac{P}{P'}$ and $\frac{M}{M'}$

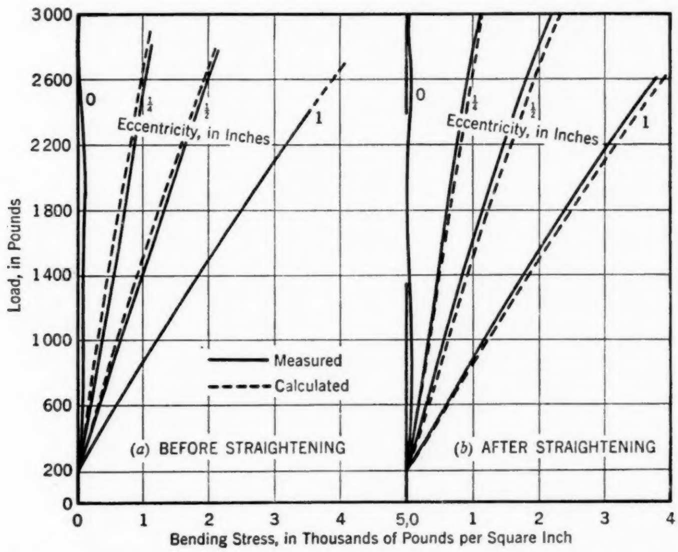


FIG. 6.—SPECIMEN 10 (69$\frac{13}{16}$ IN.); BENDING STRESSES IN THE PLANE OF THE WEB

and are plotted in Fig. 7. The applied bending moment is, of course, $M = P\,e$. The critical values of axial load (P') and bending moment[13] (M') were calculated from the following equations:

$$P' = \frac{\pi^2 E\,I}{(K\,L)^2} \quad \text{.......................(5)}$$

and

$$M' = \frac{\pi\sqrt{E\,I\,G\,J}}{K\,L}\sqrt{1 + \left(\frac{\pi\,d'}{2\,K\,L}\right)^2 \frac{E\,I}{G\,J}} \quad \text{.............(6)}$$

in which E is the modulus of elasticity, in pounds per square inch; I is the moment of inertia in a lateral direction, in inches^4; d' is the depth of the beam

[12] "Lateral Buckling of I-Section Column with Eccentric End Loads in Plane of the Web," by Bruce Johnston, *Journal of Applied Mechanics*, December, 1941, p. A-176.

[13] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 261.

(distance between the centroids of the flanges), in inches; L is the laterally unsupported length of the beam, in inches; J is the torsion factor, in inches[4]; and G is the modulus of rigidity, in pounds per square inch:

$$G = \frac{E}{2\,(1 + \mu)} \quad \text{(7)}$$

in which μ is Poisson's ratio. The K-values used in calculating P' and M' for the various specimens were those determined from the tests of the specimens under zero eccentricity. Since the exact factors causing K to be greater than 0.5 were not known, and since the values of K were all close to 0.5, no attempt was made to differentiate between the K-value defining the effective length for twisting and that defining the effective length for lateral bending. It is

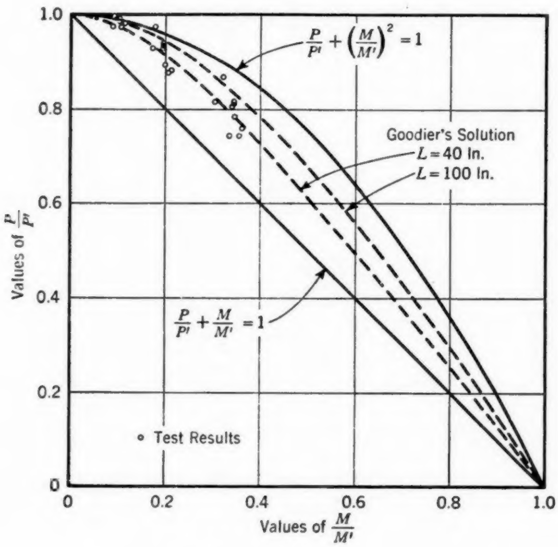


Fig. 7.—Interaction Curves of $\frac{P}{P'}$ Versus $\frac{M}{M'}$ for Eccentrically Loaded Columns

recognized, however, that for certain conditions the effective length may be different for twisting than for lateral bending. In the case of the 40-in. lengths for which a different specimen was used for each eccentricity, the K-value determined for the axially loaded case was used throughout. Section elements determined from measurements on each specimen (Table 1) were used in calculating the P'-values and the M'-values.

It will be noted in Fig. 7 that the points representing the test results generally fall below the parabola representing Eq. 4, indicating that the theoretical solution on which Eq. 3 was based gives nonconservative results.

Perhaps a more direct comparison between the test results and the theory can be obtained by comparing values of maximum load directly. For an

eccentrically loaded column Eq. 4 can be expressed as

$$P = P'\left[\frac{2}{1+\sqrt{1+4\left(\frac{eP'}{M'}\right)^2}}\right] \qquad (8)$$

Table 2 lists the critical load values calculated in accordance with Eq. 8 and the percentage variation of these values from the test results. The test results are generally lower than the calculated values, the discrepancy increasing with increasing eccentricity. The maximum difference occurred for specimen 7, a 60-in. column, when tested with 1-in. eccentricity. The calculated critical load was about 15% higher than the load at which the specimen failed.

J. N. Goodier[14] has furnished a more general solution to the problem of the buckling of a member under end load and bending, including in his derivation the hypothesis by H. Wagner describing the torsional effect of the thrust. Mr. Goodier's solution for an eccentrically loaded column of the type employed in this investigation may be expressed as

$$P = P'\left\{\frac{2}{1+\left(\rho\frac{P'}{M'}\right)^2+\sqrt{\left[1-\left(\rho\frac{P'}{M'}\right)^2\right]^2+4\left(\frac{eP'}{M'}\right)^2}}\right\} \qquad (9)$$

in which ρ is the polar radius of gyration, in inches; Eq. 9 differs from the previous expression for critical load (Eq. 8) by the introduction of the term $\left(\rho\frac{P'}{M'}\right)^2$ which is related to Mr. Wagner's critical thrust for torsional buckling. Expressed in the form of Eq. 4, Eq. 9 may be written

$$\frac{P}{P'}+\left(\frac{M}{M'}\right)^2 = 1-\frac{P}{P'}\left(\rho\frac{P'}{M'}\right)^2\left(1-\frac{P}{P'}\right) \qquad (10)$$

For a particular column specimen Eq. 10 can be plotted as a curve of $\frac{P}{P'}$ versus $\frac{M}{M'}$ similar to the parabola of Eq. 4. Fig. 7 shows typical curves obtained from Eq. 10 for the 40-in. and 100-in. specimens of this investigation. Curves for the remaining specimens would fall between the two shown. It can be seen that these curves lie below the one representing Eq. 4 and agree more closely with the test results.

Table 2 lists the critical loads for the columns tested, computed by Eq. 9, and the percentage differences between the test results and these calculated values. The values predicted by Eq. 9 vary from 2.0% less than the test value, for specimen 13 with ½-in. eccentricity, to 6.4% greater than the test value, for specimen 7 tested with a 1-in. eccentricity of load.

The difference between the test results and the theoretical values of critical load may be further reduced by consideration of the effect of deflections in the plane of the web. Eqs. 8 and 9 are based on the assumption that the moment

[14] "Flexural-Torsional Buckling of Bars of Open Section," by J. N. Goodier, *Bulletin No. 28*, Cornell Univ. Eng. Experiment Station, Ithaca, N. Y., 1942, p. 12.

of the load is constant along the length of the member. Actually the moment arm of the load in the plane of the web is equal to the eccentricity plus the deflection in this direction. The values of critical load in Table 2, Cols. 10 and 13, were computed by assuming the effective eccentricity of the load to be equal to the original eccentricity plus two thirds of the calculated center deflection in the plane of the web at failure. The value of two thirds times the maximum deflection was chosen to give an approximation of the average deflection over the length of the specimen. The variation between test results and the critical loads computed by this method ranges from -3.2% for

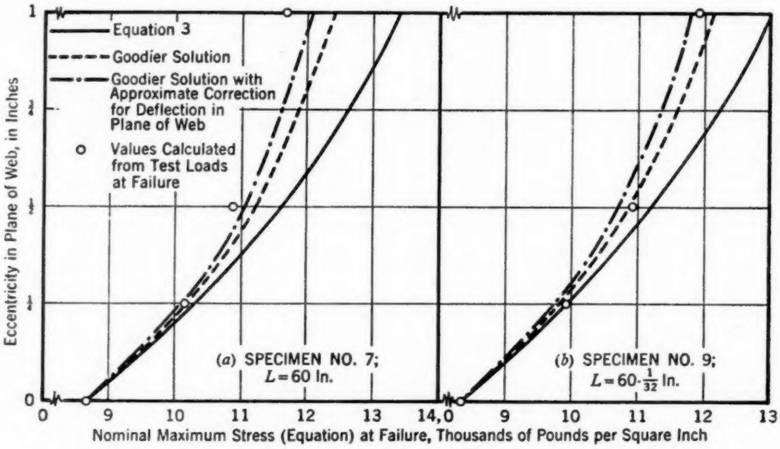


FIG. 8.—ECCENTRICITY OF LOAD VERSUS MAXIMUM STRESS AT FAILURE; 60-IN. SPECIMENS

specimen 13 tested with a 1-in. eccentricity of load to $+3.5\%$ for specimen 7 with a 1-in. eccentricity.

Since design formulas for eccentrically loaded columns are expressed in terms of the maximum allowable nominal stress—

$$f_n = \frac{P}{A} + \frac{P\,e\,c}{I} \qquad (11)$$

—the critical loads as determined by a representative few of these tests and the various theoretical solutions have been converted into maximum nominal stresses and plotted against eccentricity in Fig. 8. The agreement between test results and the various theories can be clearly seen in this illustration and in others not published.

The reason that the test results do not agree with the parabolic curve in Fig. 7 is found in the incompleteness of the theory on which the expression for the parabola is based. The test results are in satisfactory agreement with the theoretical treatment of Mr. Goodier. The writers have been unable, however, to express this solution in a form simple enough for design purposes.

The straight line in Fig. 7 represents the relationship:

$$\frac{P}{P'} + \frac{M}{M'} = 1 \qquad (12)$$

which corresponds to the equation for combined axial and bending stresses given in some specifications.[15,16] As used in Fig. 7, the P'-value is the critical load for lateral buckling under axial end load alone. The plotted points representing the test results fall above this straight line and below the parabola representing Eq. 4 which was intended to cover this type of failure.

Conclusions

The following conclusions seem justified on the basis of the foregoing results and discussion:

1. When loaded eccentrically in the plane of the web, the extruded I-section columns of this investigation failed by lateral buckling at loads that were generally lower than those predicted by the theoretical solution which was the basis for the design formula, Eq. 3. In the extreme case, the theoretical value was about 15% higher than the test load at failure.
2. The test results were in good agreement with the theoretical solution of Mr. Goodier.[14] The critical loads as predicted by the Goodier solution ranged from 2.0% less to 6.4% greater than those determined by the tests.
3. The range of variation between test results and theoretical values of critical load was reduced to $-$ 3.2% and to $+$ 3.5% by modifying the Goodier solution to take approximate account of the effect on moments of the deflections in the plane of the web at the maximum load.
4. The critical loads determined in these tests were all higher than values predicted by the straight-line formula for interaction between axial load and bending moment.

Preliminary results of additional tests undertaken to extend the range of the investigation described in this paper, confirm the solution, as modified to account for deflection in the plane of the web. They show that, for certain proportions of eccentrically loaded I-section columns, failure may occur by lateral buckling at loads considerably less than those predicted by the straight-line interaction formula, Eq. 12. This equation, however, will yield conservative results if the deflection of the member in the plane of bending is included in computing the bending moment.

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[15] "Specification for the Design, Fabrication, and Erection of Structural Steel for Buildings," AISC, New York, N. Y., February, 1946, Section 12.

[16] "Design Specifications for Bridges and Structures of Aluminum Alloy 27S-T," prepared by Leon S. Moisseiff for the Aluminum Co. of America, 1940, article 205.